Lecture - 3

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Differentiation of Products

$$y = f(x)g(x)$$

$$\frac{dy}{dx} = \frac{df(x)}{dx}g(x) + f(x)\frac{dg(x)}{dx}$$

$$y = (x^2 + 1)\sin x$$

$$\frac{dy}{dx} = \sin x\frac{d(x^2 + 1)}{dx} + (x^2 + 1)\frac{d(\sin x)}{dx}$$

$$= 2x\sin x + (x^2 + 1)\cos x$$

Differentiation of Quotients

$$y = \frac{f(x)}{g(x)}$$

Applying the product rule

$$\frac{dy}{dx} = \frac{1}{g(x)}\frac{df(x)}{dx} + f(x)\frac{d(\frac{1}{g(x)})}{dx}$$

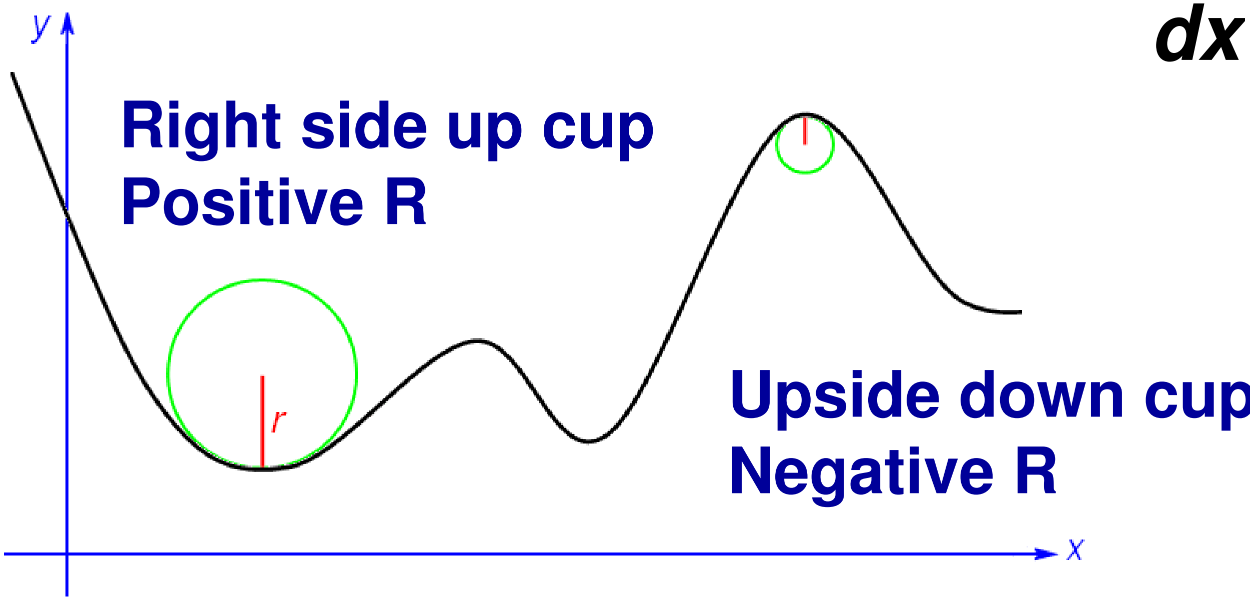
Applying the chain rule

$$= \frac{1}{g(x)}\frac{df(x)}{dx} + f(x)\left(-\frac{1}{g(x)^2}\right)\frac{dg(x)}{dx}$$

$$y = \frac{(x^2+1)}{\sin x}; \quad \frac{dy}{dx} = \frac{1}{\sin x}2x + (x^2+1)\left(-\frac{1}{\sin^2 x}\right)\cos x$$

Radius of Curvature

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) \qquad R = \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}}$$



Application in Mechanics

Path of a particle is given as

$$y = 4x^2 + x$$

When x=2, the speed of the particle is 40 m/s. Find its centripetal acceleration.

$$\frac{dy}{dx} = 8x + 1 \quad \frac{d^2y}{dx^2} = 8$$

$$at \quad x = 2$$

$$R = \frac{\{1 + (17)^2\}^{3/2}}{8} = 617.3 \quad a = \frac{v^2}{R} = \frac{(40)^2}{617.3} = 2.592 m/s^2$$

Parametric Path of Particle

Path of a particle is given by $x=4t$ and $y=3t^2$. Find the centripetal acceleration at time $t=2$.

$$\frac{dx}{dt} = 4 \quad \frac{dy}{dt} = 6t \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{6t}{4} = \frac{3t}{2}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{3t}{2}\right) = \frac{d}{dt}\left(\frac{3t}{2}\right)\frac{dt}{dx} = \frac{3}{2} \times \frac{1}{4} = \frac{3}{8}$$

Using chain rule

Parametric Path (contd.)

$$\boldsymbol{At} \quad \boldsymbol{t} = 2$$

$$\boldsymbol{v_x} = 4\boldsymbol{;}\ \boldsymbol{v_y} = 6 \times 2 = 12\boldsymbol{;}\ \boldsymbol{v}^2 = \boldsymbol{v_x}^2 + \boldsymbol{v_y}^2 = 16 + 144 = 160$$

$$\frac{\boldsymbol{dy}}{\boldsymbol{dx}} = \frac{3}{2} \times 2 = 3\boldsymbol{;}\ \boldsymbol{R} = \frac{\{1 + 9\}^{3/2}}{3/8} = 84.33$$

$$\boldsymbol{a} = \frac{\boldsymbol{v}^2}{\boldsymbol{R}} = \frac{160}{84.33} = 1.897\,\boldsymbol{m}/\boldsymbol{s}^2$$

Parametric Path (cont.)

Can we not eliminate t and write $y=f(x)$?

Yes you can, but when $x=p(t)$ and $y=q(t)$ are complex functions, eliminating t could be a lot of work. Hence, you must learn the method we just followed in the last two slides.

In-class Exercise

Path of a particle is given by $x = 4t^2+1$ and $y = 3t^{5/2}$. Find the centripetal acceleration at time $t=2$.